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CONJUGATE SYSTEMS OF SPACE CURVES WITH EQUAL
LAPLACE-DARBOUX INVARIANTS

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It is the object of this note to provide a new geometrical interpretation for the condition that the Laplace-Darboux invariants of an equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} + a \frac{\partial z}{\partial x} + b \frac{\partial z}{\partial y} + cz = 0 \quad (1)$$

be equal. These invariants are

$$h = a_x + ab - c, \quad k = b_y + ab - c, \quad (2)$$

where we have put

$$a_x = \frac{\partial a}{\partial x}, \quad b_y = \frac{\partial b}{\partial y}, \quad (3)$$

a notation which we shall employ throughout this paper.

Let $z^{(1)}, z^{(2)}, z^{(3)}, z^{(4)}$ be any four linearly independent solutions of (1), functionally independent in the sense that no two of their three ratios can be expressed as a function of the third. If $z^{(1)}, \dots, z^{(4)}$ be interpreted as the homogeneous coordinates of a point P of space, variation of the parameters x and y will cause P to generate a non-degenerate surface S which is not a plane. Let us assume further that S is not developable. Then $z^{(1)}, \dots, z^{(4)}$ will satisfy, besides (1), just one other linear homogeneous differential equation of the second order, of the form

$$z_{yy} = mz_{xx} + nz_x + pz_y + qz, \quad (4)$$

where

$$m = \frac{D''}{D}, \quad D \neq 0, \quad D'' \neq 0, \quad (5)$$

D and D'' being two of the fundamental quantities of S of the second order. The third of these quantities, D' , is equal to zero on account of the fact that the curves $x = \text{const.}$ and $y = \text{const.}$ form a conjugate system on S .

Let us now apply the Laplace transformations to the surface S , by putting

$$z_1 = z_y + az, \quad z_{-1} = z_x + bz. \quad (6)$$

These expressions determine two points P_1 and P_{-1} whose loci give rise to two further surfaces S_1 and S_{-1} . The surface S_1 is the second sheet of the focal surface of the congruence formed by the lines which are tangent to the curves $x = \text{const.}$ on S ; the surface S_{-1} is connected in similar fashion with the curves $y = \text{const.}$ on S .

Consider now the line which joins the points P_1 and P_{-1} of the 1st and -1st Laplace transformed nets. There is one such line for every point P of the original surface S . Consequently, the totality of these lines forms a *congruence*, whose developables we proceed to determine. For this purpose, let us give increments, δx and δy , to x and y . The coordinates of those points of S_1 and S_{-1} , which correspond to the point $z + z_x \delta x + z_y \delta y$ of S , will be

$$Z_1 = z_1 + \frac{\partial z_1}{\partial x} \delta x + \frac{\partial z_1}{\partial y} \delta y, \quad Z_{-1} = z_{-1} + \frac{\partial z_{-1}}{\partial x} \delta x + \frac{\partial z_{-1}}{\partial y} \delta y.$$

Now we find, making use of (1),

$$\begin{aligned} z_1 &= z_y + az, & z_{-1} &= z_x + bz, \\ \frac{\partial z_1}{\partial x} &= -bz_y + (a_x - c)z, & \frac{\partial z_{-1}}{\partial x} &= z_{xx} + bz_x + b_x z, \\ \frac{\partial z_1}{\partial y} &= z_{yy} + az_y + a_y z, & \frac{\partial z_{-1}}{\partial y} &= -az_x + (b_y - c)z. \end{aligned} \quad (7)$$

The homogenous coordinates of an arbitrary point on the line $Z_1 Z_{-1}$ will be obtained from λZ_1 and μZ_{-1} , and this expression, on account of (7), assumes the form

$$\begin{aligned} &\{\lambda [a + (a_x - c) \delta x + (a_y + q) \delta y] + \mu [b + b_x \delta x + (b_y - c) \delta y]\} z \\ &+ \{\lambda n \delta y + \mu (1 + b \delta x - a \delta y)\} z_x + \lambda \{1 - b \delta x + (a + p) \delta y\} z_y \\ &+ (\mu \delta x + \lambda m \delta y) z_{xx}. \end{aligned} \quad (8)$$

In order that such a point may also be on the line $P_1 P_{-1}$, i.e., in order that these two lines may intersect, (8) must reduce to a linear homo-

geneous combination of z_1 and z_{-1} . This will be so, if and only if λ , μ , δx , and δy can be determined subject to the two conditions

$$\lambda m \delta y + \mu \delta x = 0, \quad (9)$$

$$\lambda [h + (a_y + q - bn - a^2 - ap) \delta y] + \mu [(b_x - b^2) \delta x + k \delta y] = 0.$$

If we eliminate the ratio $\lambda : \mu$ from (9), we obtain the differential equation of the developables of the congruence, viz.:

$$h \delta x^2 + [a_y + q - bn - a^2 - ap - m(b_x - b^2)] \delta x \delta y - mk \delta y^2 = 0. \quad (10)$$

This equation may also be regarded as the differential equation of the curves which the developables of the congruence determine on S . The asymptotic curves of S are determined by

$$D \delta x^2 + D'' \delta y^2 = 0. \quad (11)$$

Therefore the curves (10) form a conjugate system on S if and only if the simultaneous invariant of (10) and (11) is equal to zero, i.e., if and only if $hD'' - mkD = 0$. But, according to (5), this reduces to $h = k$.

We have, therefore, obtained the following theorem. *Consider a net of conjugate curves on a non-developable surface S . Let P be any point of this net and let P_1 and P_{-1} be the corresponding points of the nets obtained from the given one by the 1st and -1 st Laplace transformations. Consider further the congruence of all of the lines such as $P_1 P_{-1}$. The curves on S , which correspond to the developables of this congruence, will form a conjugate system, if and only if the original net of conjugate curves has equal Laplace-Darboux invariants.*

I wish to add one further remark. Darboux¹ has given a geometric interpretation of the condition $h = k$, different from mine, by means of a certain conic in the plane of P, P_1, P_{-1} .

I have found it advisable to introduce two such conics, which coincide with each other and with that of Darboux in the case $h = k$ and only in that case. By using these conics I have been able, quite recently, to interpret geometrically the condition which Bianchi expresses by saying that a conjugate system is *isothermally conjugate*. These systems have made their appearance in so many problems of differential geometry that such a geometrical interpretation seems to me to be a matter of very great interest. I shall, however, reserve the details of this interpretation for a place, in its appropriate setting, in a paper on the general theory of congruences which I hope to present for publication before long.

¹ *Leçons sur la Théorie générale des Surfaces*, vol. 4, p. 38.